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An experimental examination of the large-eddy equilibrium hypothesis

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The large-eddy energy equilibrium hypothesis states that the largest eddies of a turbulent shear flow are in approximate energy equilibrium throughout a significant part of their lives. This hypothesis leads to a relationship between the mean rate of shear strain and the Reynolds shear stress which involves the scale of the large eddies. By assuming that the large-eddy scale is proportional to the standard deviation of the free turbulent boundary, or laminar superlayer, the validity of this hypothesis may be checked experimentally. Intermittency and mean velocity measurements made in five different two-dimensional shear flows are presented and these results, together with values calculated from Townsend's measurements in a two-dimensional wake, support the form of relationship suggested by the energy equilibrium hypothesis.

1. The energy equilibrium hypothesis

In his monograph, Townsend (1956) proposed a model for the structure of turbulent shear flow which is based upon the division of the turbulent scales into 'large eddies' comparable in size to the scale of the mean motion, and the 'remaining eddies' which contain most of the turbulent energy. The stretching of the large eddies by the mean flow is postulated as the chief mechanism whereby energy is transferred from the mean to the turbulent motion and the transfer of energy to the smaller eddies may be pictured as a repetition of the initial process but with increasingly random orientation and smaller scale.

The large-eddy equilibrium hypothesis, as its name implies, suggests that the large eddies are in approximate energy equilibrium throughout a significant part of their existence, gaining energy from the mean motion and losing energy at an equal rate to the remaining turbulent motion. An expression stating the hypothesis may be derived quite simply by equating the energy loss to the energy gain of a time-averaged turbulent motion having its own length and velocity scales (say l' and u', respectively). The assumption of these unique scales is equivalent to the assumption that the large eddy is a self-similar motion but the assumed scales need not be the same as those of the mean flow.

By assuming a particular form for the large eddies of a turbulent shear flow, Townsend (1956) was able to demonstrate the conformity of wake development, and more approximately of free-jet and boundary-layer development, to the principle of large-eddy equilibrium. Measurements by Grant (1958) have since shown, however, that the form of large eddy chosen by Townsend is not compatible with some velocity correlations in wakes or boundary layers, so that Townsend's conclusions regarding the hypothesis are somewhat doubtful.

The present experimental verification does not depend on a particular form for the large eddies, but uses values of the shear stress and large-eddy scale measured in various two-dimensional shear flows to examine the proportionality predicted by the large-eddy equilibrium hypothesis. Interest in this hypothesis has arisen because of its use, together with other assumptions, as a basis for predicting the streamwise development of certain types of turbulent shear flows (Gartshore 1965).

The energy equation for the turbulent motion may be written (in the notation used by Townsend 1956) as

$$\underbrace{\tfrac{1}{2}U_l\frac{\partial\overline{q^2}}{\partial x_l}+\overline{u_iu_l}\frac{\partial U_i}{\partial x_l}}_{\text{(A)}}+\frac{1}{2}\frac{\partial\overline{q^2u_l}}{\partial x_l}=-\frac{\partial\overline{pu_l}}{\partial x_l}+\underbrace{\nu\overline{u_i\frac{\partial^2u_i}{\partial x_l^2}}}_{\text{(B)}}. \quad (1)$$

To apply this equation to the large-eddy motion alone, only the major production and dissipation terms are considered (terms (A) and (B) respectively) and the kinematic viscosity ν is replaced by a scalar 'effective viscosity' ϵ_1 which represents the action of the smaller turbulent scales on the large eddy itself. Because the large eddy contains a relatively small fraction of the total turbulent energy, it is tempting to assume that this scalar ϵ_1 is equivalent to the usual turbulent kinematic viscosity, defined in a two-dimensional shear flow as

$$-\overline{uv}=\epsilon(\partial U/\partial y). \quad (2)$$

In any case, it will be assumed that the ratio of ϵ to ϵ_1 has a unique value. (This statement suggests a means of defining the spectral extent of the large eddies.)

Equating terms (A) and (B) of equation (1) for the large eddy in a two-dimensional shear flow, we have

$$(u')^2\,\partial U/\partial y \propto \epsilon(u')^2/(l')^2. \quad (3)$$

In terms of the shear stress $-\overline{uv}$, equation (3) becomes

$$-\overline{uv}\propto (l')^2\,(\partial U/\partial y)^2. \quad (4)$$

Equation (4) is equivalent to Prandtl's mixing length hypothesis except that l' is now identified as the scale of the large eddies in the flow.

For flows in which the mean velocity profile may be represented by a single velocity and length scale (U_0 and L_0 respectively), equation (3) becomes

$$U_0L_0/\epsilon \propto (L_0/l')^2. \quad (5)$$

Equation (5) has the same form as that of an expression derived by Townsend (§ 6.7 of his monograph) for the equilibrium condition for large eddies.

2. The basis for experimental examinations

The relatively sharp boundary to the region of turbulent flow in free turbulence is continuously wrinkled and contorted by the neighbouring turbulent motion. The amplitudes of displacement which occur in this free boundary, or viscous superlayer as it is sometimes called, are related to the scales of the eddies in the turbulent motion, and it is plausible to assume that a direct proportionality exists between the larger wrinkle amplitudes and the larger eddy scales. This proportionality need not, and probably does not extend to small eddy scales and small boundary displacements, for the lateral position of a small eddy in the flow will vary considerably from one eddy to another with varying effect on the related super-layer distortion, whereas a large eddy will have a more restricted location in the flow. The standard deviation of the viscous super-layer position is most strongly influenced by the larger displacement amplitudes and is therefore a convenient measure of the large eddy length scale.

In summary, the experimental valuation of equation (5) with which the following sections are concerned is based on the assumption that the standard deviation of the free-boundary position is directly proportional to the scale of the large eddies present in the turbulence. The standard deviation, hereafter designated by σ, has been obtained from measured distributions of the intermittency factor, as described in the next section.

All of the quantities in equation (5) have been evaluated in five different two-dimensional shear flows of which four are approximately self-preserving wall jets and one is the free jet in still air. To these results have been added values calculated from the small deficit wake, measured by Townsend (1949). In all these cases, the mean velocity profile over a major part of the flow may be presented by an expression of the form

$$U - U_1 = U_0 \exp\{-k(y/L_0)^2\},$$

where U_1 is the free-stream velocity, L_0 is the width to the point where

$$U = U_1 + \tfrac{1}{2}U_0$$

and k, in consequence of these definitions, is $\log_e 2$. (See also figure 1 for identification of symbols.)

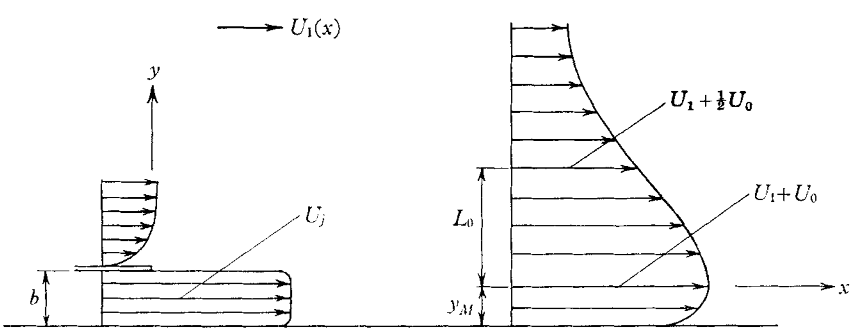

FIGURE 1. The wall jet in streaming flow; definition of symbols.

The restriction of this examination to approximately self-preserving jets and wall jets was necessitated by the requirement that the turbulence at the point of measurement should have a sufficiently long and uniform history to ensure that the large eddies of the flow were a product of equilibrium types and orientations of the mean-flow rate-of-strain tensors. It appears from Townsend's measurements of intermittency in a small deficit wake (Townsend 1949, 1956) that the intermittency at $x/d = 160$ is distinctly different from that at $x/d = 1000$, the difference being due presumably to the persistence of large eddies formed near the wake-producing body. Since no such large eddies are formed in the initial stages of a jet or wall jet, these flows approach complete similarity much more quickly than the wake, and their investigation therefore requires shorter streamwise lengths to ensure definitive results.

3. Experimental arrangements

The wall-jet measurements reported here were made in the McGill University open-circuit, blower wind tunnel which has a rectangular working section of dimensions 17 in. × 30 in. One of the 30 in. sides of this working section consists of slats which can be adjusted to bleed air from the tunnel to create an adverse pressure gradient when a screen or perforated plate is fastened over the downstream end of the section. To provide the wall jet, air is blown parallel to the free-stream direction from a slot in the other 30 in. side of the tunnel. The slot size b was varied from 0·20 to 0·30 in. and the slot Reynolds number based on the jet velocity at the slot and the slot width varied from 10,200 to 23,100 (see table 1). The tunnel settling section contains three screens with open area ratios of 0·4824, 0·4824 and 0·3855 in that streamwise order.

Mean velocities were measured with round and flattened Pitot tubes (having sharpened lips) in combination with static taps located in the wall just beneath the wall jet. A DISA, constant-temperature hot-wire anemometer was used to obtain the intermittency records.

The free jet measurements were made 100 slot widths downstream of a carefully constructed symmetrical slot, 0·180 in. in width and 24 in. in length. Large end plates were used to help maintain two dimensionality of the flow.

The intermittency was measured in two ways: (i) the visual examination of trace recordings made of the differentiated signal from the hot wire; (ii) the construction of a constant voltage or 'δ' signal which had a positive value when the hot wire was in a turbulent region but a negative value of equal magnitude at all other times (see Corrsin & Kistler 1947 for a complete description of this technique). In both cases the original signal was recorded on magnetic tape and replayed at a lower tape speed to facilitate the analysis. The first method is somewhat tedious but is attractive because of its simplicity. The discrimination level required in the second method (see Corrsin & Kistler) was adjusted by examining an oscillograph record of the 'δ' signal and the original hot-wire signal. Both methods therefore depend on the existence of obvious differences in appearance between the signal from a turbulent region and that from a non-turbulent region. These differences are usually quite clear in records which contain signifi-

No.	Flow	$\dfrac{U_1}{U_0}$	m	$\dfrac{dL_0}{dx}$	$\left(\dfrac{U_0 L_0}{\epsilon}\right)_{y=L_0}$	$\dfrac{\sigma}{L_0}$	$\dfrac{x}{b}$	$\dfrac{U_j b}{\nu}$	$\dfrac{y_M}{L_0}$	$\dfrac{x_0}{b}$
1	wall jet	0	—	0·059	55	0·310	68·5	19,200	0·161	—
2	wall jet	0·52	−0·413	0·038	49	0·336	126	15,700	0·285	13·5
3	wall jet	1·10	−0·386	0·028	36	0·403	145	12,400	0·374	−4
4	wall jet	3·04	−0·356	0·014	34	0·471	126	10,200	0·960	−27
5	free jet	0	—	0·115	31	0·400	100	23,100	0	—

TABLE 1. $U_j b/\nu$, Reynolds number based on jet velocity at slot and slot width

cant lengths of each type of flow but are particularly difficult to detect in signals which represent almost fully turbulent flow. Although differentiation of the hot-wire signal emphasizes the high-frequency fluctuations associated with the turbulent motion, it may also mask the entire signal by selectively amplifying any very high frequency noise which is present. This difficulty always arises when differentiating low-amplitude signals from a constant-temperature anemometer (which is inherently noisy) and requires the deletion of high frequencies through filter circuits or the analysis of the hot-wire signal without differentiation.

The two methods described were checked by measurements in a smooth-wall flat-plate boundary layer and in a small-deficit wake. The results corresponded, within the accuracy of the measurement, with the distributions of intermittency given by Klebanoff (1955) for the boundary layer and Townsend (1956) for the wake. For the measurements in the small deficit wake, differentiation was not used, for the reason just mentioned. The results of this preliminary investigation are summarized in table 2.

	Small deficit wake*			Boundary layer	
	$\frac{x}{d}$	$\frac{U_1 d}{\nu}$	$\frac{\sigma}{L_0}$	$\frac{U_1 \delta}{\nu}$	$\frac{\sigma}{\delta}$
Townsend (1956)	160	8400	0·50		
Klebanoff (1955)				78000	0·14
Present measurements	153	5480	0·52	43100	0·13

* The wake examined in the present measurements was formed by mounting a square rod of dimension d so that one flat face was normal to the oncoming uniform stream. Townsend's measurements were made in the wake of a circular cylinder of diameter d.

TABLE 2. Results from preliminary investigation

The intermittency factor γ is usually defined as

$$\gamma(y) = \text{prob}\,[y \leqslant Y(t)],$$

where $Y(t)$ is the instantaneous position of the boundary between turbulent and non-turbulent fluid (Corrsin & Kistler 1947). In practice, the variation of $\gamma(y)$ is nearly Gaussian and $\gamma(y)$ was approximated in the present results by a curve of Gaussian form. The probability density $P(y)$ of $Y(t)$ may be defined as

$$P(y) = -d\gamma/dy,$$

so that a convenient measure of the distortion of Y from its mean position is the standard deviation of $P(y)$ defined by σ where

$$\sigma^2 = \int_{-\infty}^{\infty} P(y)\,(y-\overline{Y})^2\,dy.$$

and where $\overline{Y}$ is the mean value of Y chosen so that

$$\int_{-\infty}^{\infty} P(y)\,(y-\overline{Y})\,dy = 0.$$

4. Results and discussion

The development of approximately self-preserving wall jets has been investigated theoretically and experimentally by Patel & Newman (1961) who showed that, for approximate self preservation, $U_1/U_0 = \text{const.}$, and $L_0 \propto x$. To provide such a flow, a pressure gradient of the form

$$dU_1/dx = U_1 m/(x+x_0)$$

is necessary, where x_0 is a virtual origin of the flow and m is a constant which may be related to U_1/U_0.

The five flows investigated are listed in table 1 with a summary of the pertinent data from each. A typical example of the streamwise variation of U_1/U_0, y_M and L_0 for a self-preserving wall jet is given in figure 2 which describes the development of wall jet no. 2 of table 1.

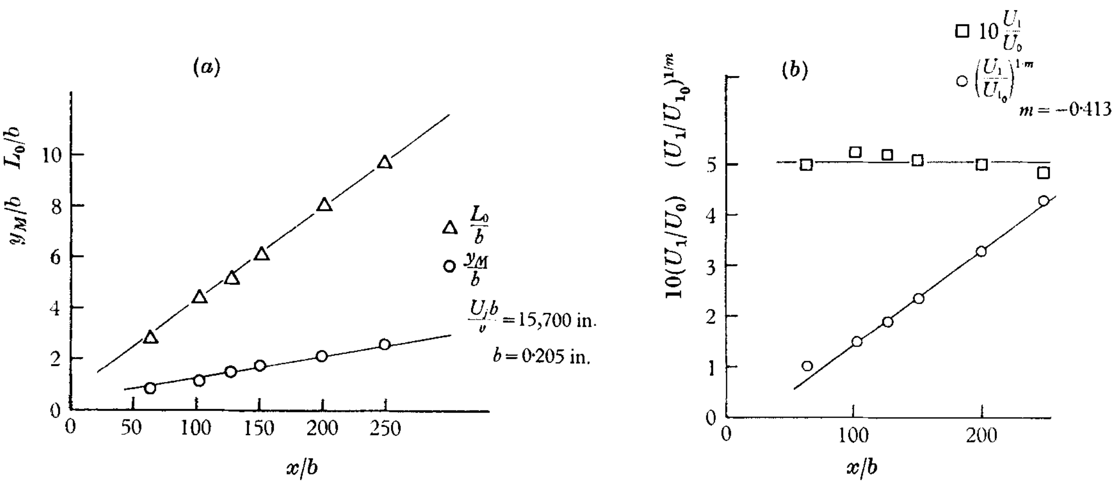

FIGURE 2(*a*). The growth of self-preserving wall-jet no. 2: $U_j b/\nu = 15700$, $b = 0{\cdot}205$ in. (*b*) The velocity decay and pressure gradient for wall jet no. 2.

The distributions of intermittency factor γ for each of the five flows are given in figure 3. Curves of the form

$$\gamma = \tfrac{1}{2}(1-\operatorname{erf}\zeta), \quad \zeta = k_1(y/L_0 - k_2)$$

have been drawn through the data, k_1 and k_2 being chosen for the best agreement with the measured points. In every case y is measured from the point of maximum velocity.

The quantity $U_0 L_0/\epsilon$ appearing in equation (5) may be calculated by assuming a mean value of ϵ over the outer part of the layer or by choosing a single non-dimensional ordinate to represent the shear stress in the outer region. For the present evaluation the latter approach has been used, $y = L_0$ being chosen as a convenient and yet typical point for the calculation. The values of $(U_0 L_0/\epsilon)_{y=L_0}$ listed in table 1 were calculated from the measured mean flow development and the streamwise boundary-layer equation of motion. (Normal turbulent stress terms were neglected.) If equation (5) is valid, a plot of $(U_0 L_0/\epsilon)_{y=L_0}$ versus $(L_0/\sigma)^2$ will appear as a straight line, since σ is assumed to be proportional to l', the large-eddy scale. Figure 4 shows these quantities for the five flows listed in table 1 and also includes values calculated from Townsend's measurements in

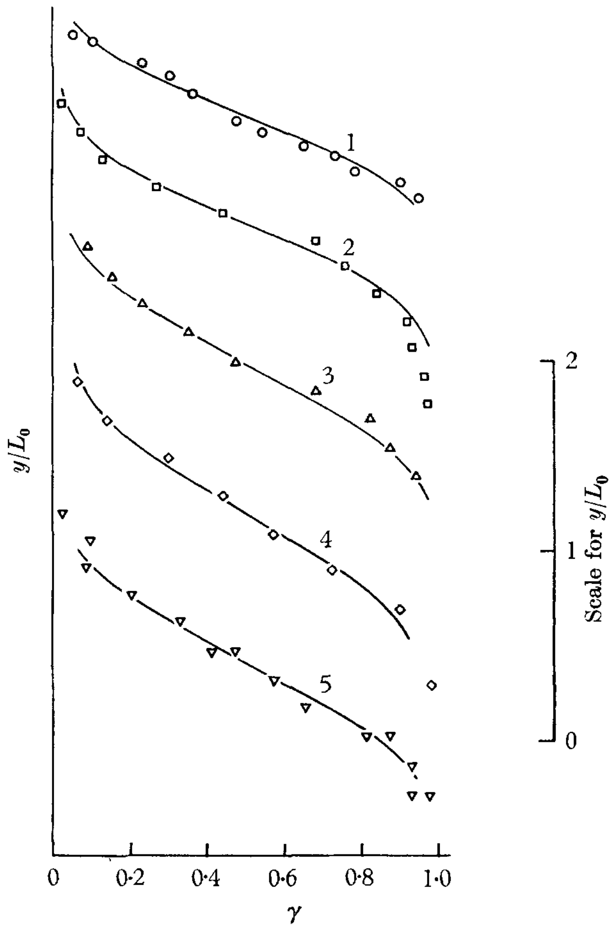

FIGURE 3. Intermittency factor distributions. Curves of the form $\gamma = \frac{1}{2}(1 - \text{erf}\,\zeta)$ have been drawn through the measured results, where $\zeta = k_1(y/L_0 - k_2)$ and k_1, k_2 are listed below:

No.	1	2	3	4	5
k_1	2·28	2·10	1·75	1·50	1·77
k_2	2·09	1·72	1·59	1·40	1·81

The numbers correspond with those of table 1. The curves may be located by means of k_2, which is the value of y/L_0 at which $\gamma = \frac{1}{2}$.

a self-preserving small-deficit wake (Townsend 1949). The numbers of the points in figure 4 correspond with those in table 1. Despite some scatter, the points in figure 4 do display the expected linear trend and thereby substantiate the form of relationship predicted by the energy equilibrium hypothesis. Flow no. 4, as may be seen from table 1, has large values of $y_M L_0$ and U_1/U_0; a significant fraction of the turbulence near $y = L_0$ has probably been produced near the wall in this case, and convected outward as in a simple boundary layer. In view of this anticipated double structure in the intermittent region it is not surprising that point 4 departs somewhat from the general trend of figure 4.

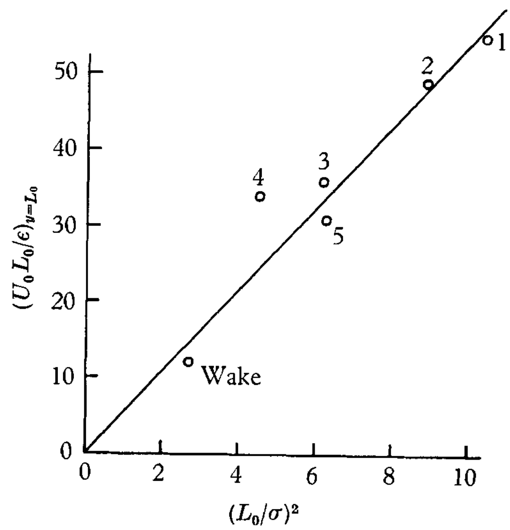

FIGURE 4. The shear stress parameter $(U_0L_0/\epsilon)_{y=L_0}$ as a function of the large-eddy scale. Numbers correspond with those of table 1.

It is interesting to note that measurements in a flat-plate boundary layer yield results which are in fair agreement with those of figure 4 once allowance has been made for the change of shape in the velocity profile. The comparison is most easily made in terms of equation (4) which, after substitution of σ for l' as before, may be written for the boundary layer as

$$-\frac{\overline{uv}}{U_1^2} \propto \left(\frac{\sigma}{\delta}\right)^2 \frac{\partial(U/U_1)}{\partial(y/\delta)}, \tag{6}$$

where δ is a nominal boundary-layer thickness. From the boundary-layer measurements of Klebanoff (1955), at the representative point $\frac{1}{2}\delta$,

$$\overline{uv}/U_1^2 = -0{\cdot}00082, \quad \partial(U/U_1)/\partial(y/\delta) = 0{\cdot}366 \quad \text{and} \quad \sigma/\delta = 0{\cdot}14.$$

Using these figures, the constant of proportionality in equation (6) would be 0·312. A similar calculation, based on recent measurements in a retarded equilibrium boundary layer (Bradshaw & Ferriss 1965) provides a value for this same constant of proportionality of approximately 0·25.

Similarly for the jets, wakes and wall jets already considered, equation (4) may be written as:

$$\epsilon/U_0L_0 \propto (\sigma/L_0)^2 k,$$

where $k = \log_e 2$. From figure 8, the relation between ϵ/U_0L_0 and $(\sigma/L_0)^2$ is approximately

$$\frac{\epsilon}{U_0L_0} = \frac{1}{5{\cdot}40}\left(\frac{\sigma}{L_0}\right)^2.$$

The constant of proportionality predicted by these values is 0·268. The agreement between these results suggests that the large eddies in the boundary layer are similar in form to those in jets or wakes.

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